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PARAMETRIC CONTROL OF THE MOTIONS OF NON-LINEAR OSCILLATORY SYSTEMS†

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A new class of control and optimization problems for the motions of oscillatory systems, utilizing adjustable variation of the parameters, is considered. Forms of the equations of motion are proposed suitable for the use of asymptotic methods of non-linear mechanics and optimal control. The basic control modes are investigated: by bang-bang-type variation of the acceleration, velocity or the value of a parameter within relatively narrow limits. Approximate time-optimal and time-quasi-optimal control laws are constructed for non-linear oscillatory systems with regulated relative equilibrium position and stiffness. Some interesting features of the motions are observed and discussed. © 2001 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

A class of controllable mechanical systems with finite constraints, which depend on adjustable parameters, is considered. To fix our ideas, we will confine our attention for the present to the case of the translational motion of a rigid body (a point mass) along a fixed curve in an inertial Cartesian system of coordinates

$$\mathbf{r} = \mathbf{r}(q, s), \quad \mathbf{r} = (x, y, z)^T, \quad q \in Q, \quad s \in S \tag{1.1}$$

where q is a generalized scalar coordinate, s is an adjustable parameter; Q and S are one-dimensional connected sets and x, y and z are sufficiently smooth functions of q and s. Differentiating expression (1.1) with respect to t, we obtain the kinetic energy K of an object of constant mass m ($m = 1$) represented as a standard quadratic form in $\dot{q}$ and $\dot{s}$

$$\begin{gathered} K = K(q, \dot{q}, s, \dot{s}) = \tfrac{1}{2}(a^2(q, s)\dot{q}^2 + 2b(q, s)\dot{q}\dot{s} + c^2(q, s)\dot{s}^2) \\ a^2 = \mathbf{r}_q'^2, \quad b = (\mathbf{r}_q', \mathbf{r}_s'), \quad c^2 = \mathbf{r}_s'^2 \end{gathered} \tag{1.2}$$

To avoid singularities in the equations of motion, we will assume that $a^2 > 0$.

It is assumed that the object is driven by potential forces described by a function W of $\mathbf{r}$ and s

$$W(\mathbf{r}, s) = W(\mathbf{r}(q, s), s) \equiv U(q, s) \tag{1.3}$$

where U is a fairly continuous function of q and s in the domain considered in (1.1). We will use the Lagrange function $L = K - U$ to obtain the equation of motion in terms of q

$$a^2\ddot{q} + aa_q'\dot{q}^2 + U_q' = -2aa_s'\dot{q}\dot{s} - b\ddot{s} + (cc_q' - b_s')\dot{s}^2 \tag{1.4}$$

Let us assume that when $s =$ const Eq. (1.4) has a stable point of rest $\dot{q} = 0$, $q_*(s)$, that is

$$U_q'(q_*(s), s) = 0, \quad U_{q^2}''(q_*(s), s) > 0, \quad s \in S$$

Then, in some neighbourhood of that point, system (1.4) will perform non-linear oscillations with constant energy E, amplitudes $q^{\pm}(E, s)$, period $T(E, s)$ and phase φ:

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$$q = q_0(\varphi, E, s), \quad E = \tfrac{1}{2}a^2\dot{q}^2 + U \geqslant \min_q U, \quad q^{\pm} = \operatorname{Arg}_q(E - U)$$

$$t = \int \frac{dq}{\dot{q}} + \text{const}, \quad T = 2\int_{q^-}^{q^+} \frac{dq}{\dot{q}^+}, \quad \omega(E, s) = \frac{2\pi}{T} \tag{1.5}$$

$$\dot{q} = \dot{q}^{\pm}(q, E, s) = \pm\sqrt{2}(E - U)^{1/2}a^{-1}, \quad \varphi = \omega t + \varphi^0$$

The constants E and φ^0 in expressions (1.5) are determined by the initial values of the Lagrange coordinates q, $\dot{q}$ and the value of the parameter s.

For applications one is naturally interested in formulating the following control problem: it is required to steer system (1.4) from an arbitrary initial state $q^0, \dot{q}^0, s^0, \dot{s}^0$ to a prescribed final state, in particular, to the state of rest $q_*(s)$, $\dot{q} = 0$, $s = s^f$, $\dot{s} = 0$, in a limited time t_f, by choosing a control action, such as $\ddot{s}$, from an admissible class of functions. In the controllable case, one formulates the problem of optimizing motions with respect to some performance index (functional) of the control, such as time-optimality, allowing for various kinds of additional conditions and restrictions [1].

In what follows, in order to use asymptotic methods for the optimal control of oscillatory systems [2–4], we will consider the situation in which the admissible value of the control function $\ddot{s}$ is in some sense small: $\ddot{s} = \varepsilon w$, where $0 < \varepsilon \ll 1$ is a small real parameter, $w \sim 1$, and the duration of the process $t_f \sim \varepsilon^{-1}$ is asymptotically long. It is required to vary the value of the total energy $E(t_f) = E^f$ in a suitable manner (by a relative magnitude of the order of unity). In this formulation, besides controllable oscillatory motions, one can also study rotational-oscillatory motions: a, b, c and U must be periodic functions of q. This natural applied approach assumes the use of the method of averaging with respect to a fast variable (with respect to the phase φ [5, 6]).

Another case which is reducible to a controllable system of type (1.4) concerns an object of a more general type, e.g., a rigid body in which a certain fixed point, such as the centre of mass, may move along the curve $\mathbf{r}(q, s)$ (1.1) and, in addition, the body takes an orientation which depends on the variables q and s. Then the angular velocity vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$, projected onto axes attached to the body, and the kinetic energy Ω of rotation have the form

$$\boldsymbol{\omega} = \boldsymbol{\Phi}(q, s)\dot{q} + \boldsymbol{\Psi}(q, s)\dot{s} \tag{1.6}$$

$$\Omega = \tfrac{1}{2}(\boldsymbol{\omega}, I\boldsymbol{\omega}) = \tfrac{1}{2}(A^2(q, s)\dot{q}^2 + 2B(q, s)\dot{q}\dot{s} + C^2(q, s)\dot{s}^2)$$

where I is the inertia tensor (matrix) of the body relative to the given point, and the vectors Φ and Ψ are defined in terms of kinematic relations between ω and variables characterizing the orientation of the body relative to the inertial system. These are usually the relations between ω and the Euler or Krylov–Euler angles, quaternions, direction cosines, etc. and their derivatives. For example, in Krylov–Euler angles we have [7]

$$\omega_1 = \dot{\alpha}\cos\beta\cos\gamma + \dot{\beta}\sin\gamma \equiv \Phi_1\dot{q} + \Psi_1\dot{s}, \quad \alpha = \alpha(q, s)$$

$$\omega_2 = -\dot{\alpha}\cos\beta\sin\gamma + \dot{\beta}\cos\gamma \equiv \Phi_2\dot{q} + \Psi_2\dot{s}, \quad \beta = \beta(q, s)$$

$$\omega_3 = \dot{\alpha}\sin\beta + \dot{\gamma} \equiv \Phi_3\dot{q} + \Psi_3\dot{s}, \quad \gamma = \gamma(q, s) \tag{1.7}$$

$$\dot{\alpha} = \alpha_q'\dot{q} + \alpha_s'\dot{s}, \quad \dot{\beta} = \beta_q'\dot{q} + \beta_s'\dot{s}, \quad \dot{\gamma} = \gamma_q'\dot{q} + \gamma_s'\dot{s}$$

Expressions for $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ similar to (1.7) are obtained if one uses the classical Euler angles or other variables [7].

If the rotating body is subject to generalized potential forces (torques) described by a function N, then, proceeding as in the case of (1.3), we obtain an expression for the potential energy of rotation M

$$N = N(\mathbf{r}, \alpha, \beta, \gamma, s) \equiv M(q, s) \tag{1.8}$$

Setting up the Lagrange function allowing for rotations for a rigid body and proceeding in the standard manner, we obtain an equation of motion in the variables q and s, similar to (1.4). The appropriate coefficients for a^2, b and c^2 and the functions K, U are sums of quantities defined by formulae (1.2), (1.6) and (1.3), (1.8).

More specific formulations of control and optimization problems for various modes of adjusting the parameter s will be given in Section 2.

2. THE HAMILTONIAN FORM OF THE EQUATIONS OF MOTION AND FORMULATION OF THE OPTIMAL CONTROL PROBLEM

The direct application of asymptotic methods to obtain approximate optimal controls for a system of type (1.4) is difficult, since the parameter s need not be a slow variable and may vary in a time $t_f \sim \varepsilon^{-1}$ by an analogous quantity. To simplify the formulation of the problem, therefore, one usually assumes that the equation does not contain s explicitly; then E and $v = \dot{s}$ will be slow, since $\dot{v} = \varepsilon w$, so that averaging methods may be used [2–4]. The required value s^f may then be obtained without changing E^f and v^f, to within the prescribed accuracy $O(\varepsilon)$.

In other specific situations the right-hand side of Eq. (1.4) may not depend explicitly on $\ddot{s}$ (when $b \equiv 0$), as is the case if the vectors $\mathbf{r}'_q$ and $\mathbf{r}'_s$ are orthogonal. It is then natural to formulate the problem of varying the variables E and s by adjusting the quantity v, $v = \varepsilon u$, $u \sim 1$, within comparatively narrow limits [2–4].

Applied control problems for dynamic objects, in which control is effected by electromechanical, pneumatic, hydraulic, etc. mechanisms, often produce models in which the velocity $\dot{s}$ is essentially a bang-bang control of comparatively small magnitude, while the acceleration $\ddot{s}$ is approximated by a generalized impulse function (Dirac δ-function) of low intensity. In this case, which is of practical importance, use of the non-linear equations of motion in their Lagrangian form (1.4) leads to considerable analytical and computational difficulties. Application of the optimality conditions of the maximum principle [1] or of the asymptotic methods of non-linear mechanics [2] is problematic. Nevertheless, it is important to note that the coefficient $b = b(q, s)$ of $\ddot{s}$ in Eq. (1.4) does not depend explicitly on $\dot{q}$ and $\dot{s}$ either, that is, the corresponding quadrature with respect to t will be defined regardless of discontinuities (jumps) in the derivatives $\dot{q}$ and $\dot{s}$.

As it turns out, these difficulties are artificial, being due to an inappropriate choice of the variables q and $\dot{q}$ to describe the dynamics, which leads to a singularity in the equations. Changing from these variables to Hamiltonian variables q and p, where p is the momentum, by a Routh transformation

$$p = \partial L / \partial \dot{q}, \quad H(q, p, s, \dot{s}) = p\dot{q} - L(q, \dot{q}, s, \dot{s}) \tag{2.1}$$

one can obtain equations of controlled motion that do not contain $\ddot{s}$. Indeed, it follows from (1.2), (1.3) and (2.1) that

$$\begin{aligned}
&\dot{q} = H'_p = a^{-2}(p - bv), \quad H = \tfrac{1}{2}a^{-2}(p - bv)^2 - \tfrac{1}{2}c^2v^2 + U \\
&\dot{p} = -H'_q = a'_q a^{-3}(p - bv)^2 + b'_q a^{-2}(p - bv)v + c'_q cv^2 - U'_q \\
&\dot{s} = v, \quad v = \varepsilon u, \quad u^- \leqslant u \leqslant u^+, \quad u^- < 0, \quad u^+ > 0
\end{aligned} \tag{2.2}$$

In this equation and in (2.1) H is the Hamiltonian of the system, which defines the first integral (energy) when $\varepsilon = 0$, i.e. $\dot{s} \equiv 0$. It is assumed in the equations of motion (2.2) that $0 < \varepsilon \ll 1$ is a small parameter, while the control u may take values in the range $[u^-, u^+]$. As before (see Section 1), considering the uncontrolled system (2.2) (i.e. where $\varepsilon = 0$), one obtains relations of the type (1.5), which define oscillatory motions. For the slow variables E and s and sufficiently small values of ε, the control and optimization problems are formulated in the standard way [2–4]. The equations of the controlled motion in standard form are

$$\begin{aligned}
&\dot{E} = pa^{-5}(b'_q a - a'_q b)(p - bv)v + pa^{-2}c'_q cv^2 - a^{-2}bU'_q v - \\
&-p^2a^{-3}a'_s v + U'_s v \equiv \varepsilon Fu + O(\varepsilon^2), \quad E = \tfrac{1}{2}a^{-2}p^2 + U \\
&\dot{s} = v, \quad v = \varepsilon u, \quad u^- \leqslant u \leqslant u^+
\end{aligned} \tag{2.3}$$

The variable p in (2.3) is related to E, q and s (and v) by formulae (2.1) and (1.5). This enables us to average the right-hand sides of the controlled system with respect to the phase φ (after determining u) by integrating with respect to q, on the basis of the integrals of the unperturbed system [2–4]. For that reason, explicit expressions for the perturbed phase φ will not be needed in what follows; moreover, this phase is not determined with adequate accuracy (see below).

Equations of the type (2.3) may also be obtained and used in the case of the control $\ddot{s} = \varepsilon w$, when Eq. (1.4) does not contain the variable s explicitly. One then takes $v = \dot{s}$ as the slow variable. We recall that the variable s is not slow; when a, b, c and U depend explicitly on s, this essentially complicates the use of the asymptotic methods of [3, 4]. Thus, if $K = K(q, \dot{q}, v)$, $U = U(q)$, we obtain the following equations for E and υ

$$\dot{E} = -\varepsilon(a^{-2}b(p - b\upsilon) + c^2\upsilon)w \equiv \varepsilon G w, \quad \dot{\upsilon} = \varepsilon w, \quad w^- \leq w \leq w^+, \quad E = H(q, p, \upsilon) \tag{2.4}$$

Note that the terms in (1.4) that contain a_s', b_s' vanish. The function K does not depend on s if a^2, b and c^2 do not depend on s; this is the case, for example, if $\mathbf{r} = \boldsymbol{\rho}(q) + \mathbf{n}s$, where $\mathbf{n}$ is a constant vector; in applications it frequently happens that $\mathbf{r} = \boldsymbol{\rho}(q) + \boldsymbol{\nu}(s)$. The variable p in (2.4) is expressed in terms of $\dot{q}$, q, υ, while the generalized velocity $\dot{q}$ is related to the variables E, q and υ in accordance with relations (2.2), which makes it possible, as in the case of Eqs (2.3), to average the equations of controlled motion on the basis of the first integral, by quadrature with respect to q.

We can now formulate the problems of the time-optimal variation of E and of the variables s or υ, respectively, or systems (2.3) and (2.4), and investigate them in an approximate setting:

$$\begin{gathered} E(t_f) = E^f, \quad s(t_f) = s^f, \quad t_f \to \min_u, \quad u^- \leq u \leq u^+ \\ E(t_f) = E^f, \quad v(t_f) = v^f, \quad t_f \to \min_w, \quad w^- \leq w \leq w^+ \end{gathered} \tag{2.5}$$

It is proposed to construct an optimum synthesis for u as a function of q, $\dot{q}(p)$, s or for w as a function of q, $\dot{q}(p)$, υ, and also to compute the averaged slow variables E and s or E and v (see below). These formulations of the optimal control problem are fundamental. They are of particular interest for applications.

In practice, however, one may have a situation in which the kinetic energy $K(q, \dot{q})$ is either independent of s and $\dot{s}$, that is, $\mathbf{r} = \mathbf{r}(q)$, or corresponds to the case $\mathbf{r} = \mathbf{n}(q + s) + \mathbf{r}_0$, where $\mathbf{n}$ and $\mathbf{r}_0$ are constant vectors (see Section 4). One either allows U to depend on the parameter s or introduces the variable $l = q + s$ [2–4]. Then, within the framework of the approach proposed here, one can formulate an optimal control problem for oscillations of the system by bang-bang type variation of the parameter s within certain narrow limits. The values of the parameter s may determine the equilibrium positions or the coefficient of elasticity [3, 4] (see Section 4).

The equations of motion in the variables q and $\dot{q}$ or l and $\dot{l}$ have the form (1.4) with the right-hand side identically zero. The equations in Hamiltonian form (2.2) do not contain the functions b and c and their derivatives. For the slow variable E, corresponding to a certain given value of $s = s^*$, the time-optimal problem has the form

$$\begin{gathered} \dot{E} = (U_q'(q, s^*) - U_q'(q, s))\dot{q} = -\varepsilon U_{qs}''(q, s^*)\dot{q}\sigma + O(\varepsilon^2) \\ E = \tfrac{1}{2}a^2(q)\dot{q}^2 + U(q, s^*), \quad s = s^* + \varepsilon\sigma \, (\dot{E} \equiv 0, \ s = s^*) \\ E(t_f) = E^f, \quad t_f \to \min_\sigma, \quad \sigma^- \leq \sigma \leq \sigma^+ \end{gathered} \tag{2.6}$$

We may assume without loss of generality that $\sigma^\pm = \pm 1$. This formulation of the problem permits a relatively small, almost instantaneous, change in the value of the restoring force $-U_q'$. It will be interesting to compare the different modes obtained when the control is effected through $\ddot{s}$, $\dot{s}$ and s.

3. SCHEMES FOR THE APPROXIMATE SOLUTION OF TIME-OPTIMAL PROBLEMS

Applying the general approach of the maximum principle [1] and the asymptotic technique of [2–4] to systems (2.3), (2.4) and (2.6), we introduce conjugate variables, form the Hamiltonians of the control problems and maximize them with respect to u, w and σ in the appropriate intervals. The maximum values of the Hamiltonian functions (Pontryagin functions) Π are averaged over the phase φ, relative to which they are 2π-periodic. This yields expressions for the Pontryagin functions in the first approximation with respect to ε, as well as canonical (Hamiltonian) equations that do not contain fast variables. The averaged equations of the maximum principle admit of a first integral and the introduction of a slow variable $\tau = \varepsilon t$, which varies in a relatively short interval $\tau_f \sim 1$. This constitutes a fundamental simplification in the analytical or numerical solution of the boundary-value problems of the maximum principle [2–4].

We will first consider the mode of control by adjustable variation of the velocity $\dot{s} = \varepsilon u$ according to (2.3) and (2.5). The optimal control u^*, the maximum value Π^* of the Hamiltonian and its average $\varepsilon\Pi_0$ are described by the following expressions (angular brackets denote averaging)

$$u^* = d^+ + d^- \operatorname{sign}\Psi,\ \ \Psi \equiv \xi F + \eta,\ \ d^\pm = \tfrac{1}{2}(u^+ \pm u^-),\ \ d^- > 0,\ \ d^- > |d^+|$$

$$\Pi^* = \varepsilon(d^+\Psi + d^-|\Psi|),\ \ \langle\Pi^*\rangle = \varepsilon\Pi_0 = \varepsilon(d^+\langle\Psi\rangle + d^-\langle|\Psi|\rangle) \tag{3.1}$$

$$\langle\Pi^*\rangle = \frac{1}{T}\int_0^T \Pi^* dt = \frac{1}{2\pi}\int_0^{2\pi}\Pi^* d\varphi = \frac{1}{T}\oint \Pi^*\frac{dq}{\dot q} = \frac{\sqrt{2}}{T}\int_{q^-}^{q^+}\Pi^*\frac{a\,dq}{\sqrt{E-U}}$$

where ξ and η are slow variables, conjugates of E and s, respectively; the quantities $T(E, s)$, $q^\pm(E, s)$ are defined as in (1.5) and are found by investigating the uncontrolled motion, that is, the free oscillations of the system when s = const. We may assume without loss of generality that $d^- = 1$ in (3.1); then $|d^+| < 1$. The typical situation in applications is usually $d^+ = 0$, that is, $|u| \leqslant u_0$ ($u_0 = 1$), and the first terms in the functions u, Π^* and Π_0 are dropped.

Using expressions (3.1), we obtain the averaged boundary-value problem of the maximum principle for $\Pi_0(E, s, \xi, \eta)$, retaining the old notation

$$E^\cdot = \Pi'_{0\xi},\ \ s^\cdot = \Pi'_{0\eta},\ \ \xi^\cdot = -\Pi'_{0E},\ \ \eta^\cdot = -\Pi'_{0s} \tag{3.2}$$

$$E(0) = E^0,\ \ E(\tau_f) = E^f,\ \ s(0) = s^0,\ \ s(\tau_f) = s^f,\ \ \Pi_0 = 1$$

Problem (3.2) contains five unknown parameters (including τ_f) and the same number of boundary conditions for determining them. In the general case, it may be solved by numerical methods. In some special cases the expression for Π_0 has a definite structure which enables the analytical methods of [2–4] to be used.

We now consider the mode of control by acceleration variation: $\dot v = \varepsilon w$, $w^- \leqslant w \leqslant w^+$ (see (2.4) and (2.5)). Expressions of type (3.1) for constructing the averaged boundary-value problem, similar to (3.2), have the same form with the following changes

$$u^* \to w^*,\ \ \Psi = \xi G + \eta,\ \ s \to v,\ \ \dot q = \pm\sqrt{2}a(E + \tfrac{1}{2}c^2 v^2 - U)^{1/2}$$

$$q^\pm(E, v) = \operatorname{Arg}_q(E + \tfrac{1}{2}c^2(q)v^2 - U(q)) \tag{3.3}$$

where ξ and η are slow variables conjugate to E and v. The structure of the averaged boundary-value problem for determining an approximate solution of the time-optimal problem with respect to the variables E and v is identical with (3.2), but the function G in (2.4) is given by an expression that is much simpler than F (2.3). In particular, if $\mathbf{r} = \boldsymbol{\rho}(q) + \mathbf{n}s$, where n is the unit vector, then $b = (\rho', \mathbf{n})$, $c^2 = 1$, and the momentum $p = a^2\dot q + bv$, where the increment bv is the projection of the velocity $\mathbf{n}v$ onto the direction $\boldsymbol{\rho}'$ (apart from the factor a).

The mode of control by adjustable variation of the parameter $s = s^* + \varepsilon\sigma$, $|\sigma| \leqslant 1$, which occurs only in the expression for the restoring force $U'_q(q, s)$ (2.6), leads to simpler relations than (3.1)–(3.3), since there are no slow variables of type s and v or the corresponding conjugate variables. As a result, one obtains explicit expressions for the control σ^* and finite formulae for E and τ_f

$$\sigma^* = -\operatorname{sign}(\Delta E U''_{qs}\dot q),\ \ E^\cdot = D(E)\operatorname{sign}\Delta E,\ \ \Delta E = E^f - E$$

$$\tau = \tau(E, E^0, E^f) = \operatorname{sign}\Delta E^0\int_{E_0}^{E}\frac{d\chi}{D(\chi)},\ \ D(E) = \frac{1}{T}\int_0^T |U''_{qs}\dot q|\,dt = \frac{2}{T}\int_{q^-}^{q^+}|U''_{qs}|\,dq \tag{3.4}$$

$$\tau_f = \tau_f(E^0, E^f) = \tau(E^f, E^0, E^f),\ \ q^\pm(E) = \operatorname{Arg}_q(E - U(q, s^*))$$

By determining the parts of the interval $q^- \leqslant q \leqslant q^+$ in which the sign of the function $U''_{qs}(q, s^*)$ is invariable, one can obtain an explicit expression for $D(E)$ as a function of U'_s (see Section 4.1). The period $T(E)$ of the oscillations in (3.4) is determined in accordance with (1.5).

It should be noted that for essentially non-linear systems, when the oscillation period depends on the energy and the slow variables, the control must be obtained by synthesis, since the fast variables, including the phase φ, are computed with an error $O(1)$ [2–4]. After the conjugate variables ξ and η have been found as functions of τ, E^0 and s^0 or v^0, one substitutes $\tau \to 0$, $E^0 \to E$ and $s^0 \to s$ or $v^0 \to v$, which leads to a feedback control, i.e. one depending on $q, \dot q$ (or p) and s or v, which are assumed to be measurable.

4. SPECIFIC MODELS OF NON-LINEAR OSCILLATORS WITH ADJUSTABLE PARAMETERS

4.1. *A one-dimensional oscillator with controlled equilibrium position*
Consider the linear motion of a body of mass $m = 1$ driven by a non-linear elastic force in the neighbourhood of a stable equilibrium position (see Sections 1 and 2). Three control modes can be considered

$$\begin{aligned}
&x = q+s,\ \ K = \tfrac{1}{2}(\dot q+\dot s)^2,\ \ U = U(q),\ \ U(0) = U'(0) = \ldots = U^{(2n-1)}(0) = 0\\
&\ddot q + U'(q) = -\varepsilon w,\ \ \dot v = \varepsilon w\ \ (\dot s = v),\ \ w^- \leqslant w \leqslant w^+\\
&\dot q = p - \varepsilon u,\ \ \dot p = -U'(q),\ \ \dot s = \varepsilon u,\ \ u^- \leqslant u \leqslant u^+\\
&\ddot x + U_q'(x-s) = 0,\ \ s = s^* + \varepsilon\sigma,\ \ |\sigma| \leqslant 1\ \ (U_{(q)}^{(2n)} > 0,\ \ q\in Q)
\end{aligned}\tag{4.1}$$

According to (4.1), $q = 0, \dot q = 0$ is a point of rest of the system if $\varepsilon = 0$. Control by acceleration εw enables the system to be steered to a relative state of rest or oscillations with prescribed energy E^f (or amplitudes $q^{\pm}(E)$) and the desired velocity v^f of displacement as a whole (see (2.4), (2.5)). If the velocity $\dot s = \varepsilon u$ of displacement of the equilibrium position can be adjusted, then a solution in variables q and p can be found for the problem of steering the system to position E^f, s^f (see (2.3) and (2.5)). Finally, if the position s is the adjustable parameter ($\varepsilon\sigma$ is the control), then the oscillator will be brought to a state E^f (see (2.6)). Thus, the models (4.1) of controlled motions of the oscillator enable one to consider the three control modes just described: by acceleration, velocity and displacement of the equilibrium position. By (3.1)–(3.5), the optimal control problems for slow variables generally require the use of numerical methods to solve the appropriate boundary-value problems of the maximum principle [2–4].

Control by velocity variation has been studied fairly thoroughly in a quasi-linear setting by analytical methods [3, 4]. It has been established, by changing to the amplitude A and phase φ, that the conjugates ξ and η of the corresponding slow variables A and s do not vary with time. The desired solution of the problem is completely determined in terms of the root $\varkappa$ of a transcendental equation; we have

$$\begin{aligned}
&\Delta s/|\Delta A| = [(\pi/2)(d^+/d^-) + \operatorname{sign}\Delta A \arcsin\varkappa]/\sqrt{1-\varkappa^2},\ \ \varkappa = \eta/\xi,\ \ |\varkappa|<1\\
&\tau_f = (\pi/(2d^-))|\Delta A|/\sqrt{1-\varkappa^2},\ \ \Delta A = A^f - A \neq 0,\ \ \Delta s = s^f - s\\
&u^* = d^+ + d^- \operatorname{sign}(\Delta A(\varkappa - q/A)),\ \ A = A^0 + \Delta A^0\tau/\tau_f^0,\ \ s = s^0 + \Delta s^0\tau/\tau_f^0\\
&q = A\sin\varphi,\ \ \dot q = A\cos\varphi,\ \ p = \dot q + \varepsilon u^*,\ \ \varphi = t + O(1)
\end{aligned}\tag{4.2}$$

Here the approximate optimum control u^* is represented as a feedback; to construct a programmed control the phase φ must be constructed more accurately [3]. The unknown $\varkappa$ is determined from the first equation of (4.2), either numerically or graphically, in terms of the given quantities ΔA, Δs and d^+/d^-.

In the non-linear case, as already remarked, one has to resort to numerical methods to solve a non-linear boundary-value problem of type (3.2), so that considerable difficulties are involved. One possible procedure, nevertheless, is quasi-optimal subdivision of the control process, first varying the energy (amplitudes) of the relative oscillations in an optimal manner, and then moving the oscillator as quickly as possible to the desired position. Estimates indicate that the relative increase in time is small.

We will briefly describe the solution of the time-optimal control problem when the energy of oscillations is varied. By (2.3), (3.1) and (3.2), we have

$$u^* = d^+ + d^- \operatorname{sign}(\Delta E F(q)), \quad F = -U'(q), \quad \Delta E = E^f - E$$
$$\tau = \tau(E, E^0, E^f) = \operatorname{sign}\Delta E^0 \frac{1}{d^-} \int_{E^0}^{E} \frac{d\chi}{F_0(\chi)}, \quad \tau_f = \tau(E^f, E^0, E^f) \tag{4.3}$$
$$F_0(E) = \frac{1}{T}\int_0^T |F|\,dt = \frac{\sqrt{2}}{T}\int_{q^-}^{q^+} \frac{|F|}{\sqrt{E-U}}\,dq = 4\sqrt{2}\frac{\sqrt{E}}{T(E)}$$

(The computation of the mean $F_0(E)$ used an expression for $F(q)$ from [2–4].) It follows from (4.3) that τ_f will be finite for $E^f = 0$ if $F_0(E) \sim E^\delta$, $\delta < 1$, that is, if $\gamma > -1/2$, where γ is the exponent on the right of the estimate $T(E) \sim E^\gamma$ for small E. For example, if $U = \frac{1}{2}q^2$, then $T = 2\pi$, and we obtain $\gamma = 0 > -1/2$; then $\tau_f = \pi\sqrt{E^0/2}/d^-$. The treatment of controllability when $E^0 = 0$ is analogous. In the case of a power non-linearity $U = k|q|^\rho$, $\rho > 0$, we obtain the formulae

$$T(E) = \theta(k,\rho)E^{1/\rho - 1/2}, \quad \theta = (2\sqrt{2\pi}/\rho)\Gamma(1/\rho)\Gamma^{-1}(\tfrac{1}{2} + 1/\rho)k^{-1/\rho} \tag{4.4}$$
$$E(\tau) = [(E^0)^{1/\rho} + ((E^f)^{1/\rho} - (E^0)^{1/\rho})\tau/\tau_f]^\rho, \quad \tau_f = (\rho\theta/(4\sqrt{2}d^-))\,|(E^f)^{1/\rho} - (E^0)^{1/\rho}|$$

where Γ is the Euler gamma-function. For a power function U, it follows from (4.4) that τ_f will always be finite for $E^{0,f} = 0$ (damping or build-up of oscillations).

Note that sign $Fq = -1$, that is, the control "tends" to shift the equilibrium point toward a deviation that will reduce the restoring force. Conversely, to increase the energy of relative oscillations (oscillation build-up), the control merely has to increase the restoring force. Thus, the control of relative oscillations has the character of a resonance action. The oscillator is moved to the desired position s^f by applying a constant control $u^{**} = d^+ + d^-$ sign Δs, which does not affect the value of the energy of relative oscillations within the range of accuracy considered here (error $O(\varepsilon)$). If $d^+ = 0$, it may be preferable to move the oscillator to the final point s^f and then change the energy of the oscillations. The control will be globally time-optimal if the slow variables are brought simultaneously to the final point E^f, s^f, which means solving a boundary-value problem of type (3.2).

Let us consider the process of controlling the motions of the oscillator by controlling the acceleration of the motion of the equilibrium position. By (4.1), we obtain the following equations for the energy E of relative oscillations and the velocity v (see (2.4) and (2.5))

$$\dot{E} = -\varepsilon\dot{q}w, \quad \dot{v} = \varepsilon w, \quad w^- \leqslant w \leqslant w^+, \quad w^\pm \gtrless 0 \tag{4.5}$$

To solve a boundary-value problem of type (3.2), (3.3) for system (4.5) in the general case of non-linear oscillations, one must resort to numerical methods. In a quasi-linear treatment, the solution of the time-optimal problem is constructed in terms of the variables A and v just as in the case previously considered of control by velocity. We recall that $A = \sqrt{2E}$ is the amplitude of relative quasi-linear oscillations. The difference is that in formula (4.2) for the control u^* ($u^* \to w^*$) one must substitute $\sin\varphi \to \cos\varphi$. This means that the control, i.e. the acceleration of the equilibrium position, must have the same direction as the velocity of relative motion of the oscillator for the problem of reducing energy (damping of oscillations), that is, the "inertia force" is opposed to the velocity of the relative oscillations. Conversely, the acceleration of the equilibrium position will be opposed to the velocity, and the "inertia force" in the same direction, for the problem of oscillation build-up.

Time-optimal control of non-linear oscillations ignoring the velocity v is achieved in a manner similar to that considered above (see (4.3) and (4.4))

$$w^* = d^+ - d^- \operatorname{sign}(\Delta E\dot{q}), \quad \Delta E = E^f - E, \quad \dot{q} = \pm\sqrt{2}(E - U(q))^{1/2}$$
$$\tau = \tau(E, E^0, E^f) = \operatorname{sign}\Delta E \frac{1}{d^-}\int_{E^0}^{E} \frac{d\chi}{\langle|\dot{q}|\rangle}, \quad \tau_f = \tau(E^f, E^0, E^f) \tag{4.6}$$
$$\langle|\dot{q}|\rangle = \frac{1}{T}\int_0^T |\dot{q}|\,dt = \frac{2}{T(E)}(q^+(E) - q^-(E)), \quad q^\pm = \operatorname{Arg}_q(E - U)$$

In particular, if the function $U(q)$ has the form $U = k\,|q|^\rho$, the oscillations will take place between symmetric limits $q^\pm = \pm\,(E/k)^{1/\rho}$, and the function $T(E)$ is as defined in (4.4). Thus, the mean

$\langle|\dot{q}|\rangle \sim E^{1/2}$ does not depend on ρ, indicating that the energy $E(\tau)$ varies quadratically; see the formula for E in (4.4), in which one puts $\rho = 2$, while in the formula for τ_f the factor is $\frac{1}{2}\theta(k,\rho)(d^{-}k^{1/\rho})^{-1}$, where $\rho > 0$ is any number. After the required change in the energy of the relative oscillations, achieved by applying a constant control $w^{**} = d^{+} + d^{-}\,\mathrm{sign}\Delta v$, the desired value of the velocity of the oscillator as a whole is obtained. As remarked, the sequence of stages of the control process may be changed and modified appropriately.

Consider the problem of controlling the oscillations of an oscillator by regulating the equilibrium position; on the basis of the third model of (4.1). By (2.6), the equation for the change in energy E in the first approximation with respect to ε is

$$\dot{E} = \varepsilon U''_{q^2}(x - s^*)\dot{x}\sigma,\ |\sigma| \leq 1,\ E(0) = E^0,\ E(t_f) = E^f$$

$$E = \tfrac{1}{2}\dot{x}^2 + U(x - s^*),\ U(0) = U'(0) = \ldots = U^{(2n-1)}(0) = 0,\ U^{(2n)} > 0 \tag{4.7}$$

$$x^- \leq x \leq x^+,\ x^{\pm} = s^* + \zeta^{\pm}(E),\ \zeta^{\pm} = \mathrm{Arg}_{\zeta}(E - U(\zeta))$$

Using standard methods and the relations for the more general model (2.6), we obtain the desired solution to the time-optimal control problem

$$\sigma^* = \mathrm{sign}(\Delta E U''_{q^2}(x - s^*)\dot{x}),\ \Delta E = E^f - E < 0\ (\Delta E > 0)$$

$$\tau = \tau(E, E^0, E^f) = \mathrm{sing}\,\Delta E^0 \int_{E^0}^{E} \frac{d\chi}{D(\chi)},\ \tau_f = \tau(E^f, E^0, E^f) \tag{4.8}$$

$$D(E) = \frac{1}{T}\int_0^T |U''_{q^2}\dot{x}|\,dt = \frac{2}{T(E)}(U'_q(\zeta^+) - U'_q(\zeta^-))$$

It follows from (4.8) that displacements $\varepsilon\sigma$, $\sigma = \pm 1$, of the equilibrium point s relative to the fixed value of s^* occur twice in each period of oscillations, in a direction opposite to the velocity of deviation of the oscillator, and this happens exactly when the velocity changes sign. The time τ_f will be finite for $E^f = 0$ (or $E^0 = 0$) if it is true that $D(E) \sim E^{\delta}$, $\delta < 1$, for small E. In the case of quasi-linear oscillations we have the estimate $D \sim E^{1/2}$. For a power non-linearity $U = k|q|^{\rho}$, $\rho > 0$, we obtain $\zeta^{\pm} = \pm (E/k)^{1/\rho}$, that is, one has an estimate $D \sim E^{3/2-2/\rho}$, analysis of which implies the inequality $\rho < 4$ as a precondition for $\delta < 1$. Thus, in the case of a cubic non-linearity ($U \sim q^4$) or a non-linearity of higher order, bringing the oscillator to a state of rest $E^f = 0$ (or removing it from the state $E^0 = 0$) requires an "infinite time" τ_f. The functions $E(\tau)$ and the response time $\tau(E^0, E^f)$ for a non-linearity of the power type are determined by standard methods; they are obtained in an explicit form similar to (4.4), with appropriate changes in the parameters.

Thus, the modes presented above for controlling the oscillations of an oscillator – by velocity, acceleration and equilibrium position – have interesting and instructive properties which are quite difficult to establish on the basis of intuitive conceptions.

4.2. Models of oscillatory mechanical systems with controlled equilibrium positions

Previously we investigated time-optimal control problems for the simplest model – a point or rigid body moving along a straight line and subject to an elastic restoring force with controlled equilibrium position. It should be noted that the non-linear force $-U'(q)$ may be obtained by imposing ideal constraints on a system with linear elastic elements [3, 4]. Consider a spring with linear characteristic, at one of whose ends there is a body of mass m which can move without friction along a straight line – the x axis. The other end of the spring may be displaced in a adjustable manner along a parallel straight line s (Fig. 1). Suppose the distance between the straight lines is d, and the length of the non-deformable spring $l = l_0$. Then the following formulae hold for the kinetic and potential energies of the system

$$K = \tfrac{1}{2}m\dot{x}^2,\ U = \tfrac{1}{2}k\delta^2,\ \delta = l - l_0,\ l = (d^2 + q^2)^{1/2},\ q = x - s \tag{4.9}$$

The variables and parameters of the problem may be normalized so that $m = k = d = 1$. If $d > l_0$, the spring will be stretched for all q; then $q = 0$ is a stable equilibrium position of the "centre" type. Time-optimal problems analogous to those studied in Section 4.1 can be formulated and their approximate solutions found. For small values of $|q|$ (relatively large d) the oscillations of the system will be

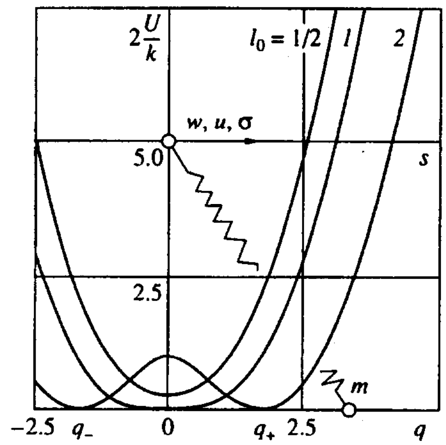

Fig. 1

nearly linear, and the control problem may be investigated in a quasi-linear setting. For large deviations $|q|$ the oscillations are essentially non-linear and separate quasi-optimal controls may be constructed for the problem.

When $d < l_0$, we have $\delta \gtrless 0$, depending on q, with $q = 0$ an unstable equilibrium position of the body, of the "saddle" type. In addition, there are two symmetric stable points of the "centre" type: $q_\pm = \pm (l_0^2 - d^2)^{1/2}$. In this case optimal control problems may be investigated approximately both in the neighbourhood of each stable point and "in the large," taking into account passage through the saddle point (separatrix crossing), corresponding to the compressed state of the spring.

The case $d = l_0$ is critical, since the function $U(q)$ contains no quadratic terms: $U = {}^1\!/_8 kd^{-2}q^4 + O(q^6)$, and the restoring force $-U'$ is a cubic function for small $|q|$. This situation leads to essentially non-linear oscillations, whose approximate control may be achieved by the methods described above.

We now consider another plane model of one-dimensional oscillations, as illustrated in Fig. 2. Suppose a point of mass m is moving without friction along a circular guide of radius R. It is driven by a non-linear elastic force (spring) with potential $U(q)$ (in particular, $U = {}^1\!/_2 kg^2$), where $q = R(\psi - \vartheta)$, where ψ is the angular coordinate of the point and ϑ is the angular coordinate of the other end of the spring or the equilibrium position of the restoring force. Then, taking the geometrical constraints into account, we obtain the following formulae (see (1.1) and (1.2))

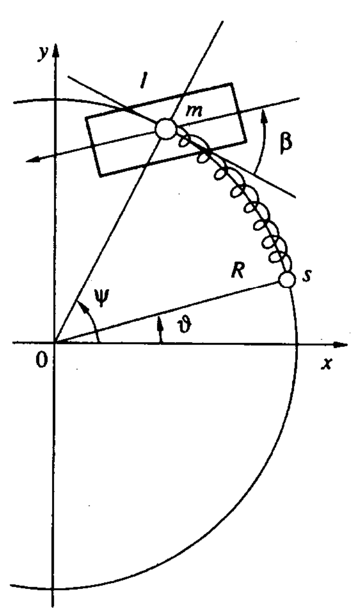

Fig. 2

$$K = \tfrac{1}{2} mR^2\dot\psi^2, U = U(q), \ \ x = R\cos\psi, \ \ y = R\sin\psi \tag{4.10}$$

which lead to the model considered above, of type (4.1), that is, the control problems for motion around a circle and along a straight line are equivalent. If a fixed point of a rigid body (such as its centre of mass) is moving around a circle, preserving the orientation of the body, then formulae (4.10) remain valid. One can also consider a more complicated function $U(q, s)$, $s = R\vartheta$.

Suppose the body is subject to a constraint: while its centre of mass is moving around the circle, the body itself is revolving about its centre of mass, while maintaining its direction relative to the tangent: $\alpha = \pi/2 + \psi + \beta$, where β = const and α is the angle of rotation of the body. Then the total kinetic energy, including that of rotation, is

$$K = \tfrac{1}{2} mR^2\dot\psi^2 + \tfrac{1}{2} I\dot\alpha^2 = \tfrac{1}{2}(mR^2 + I)\dot\psi^2$$

which is identical with (4.10), apart from a constant factor. This property may be used locally for arbitrary fairly continuous curves and is therefore of interest for applications.

4.3. An oscillator with controlled stiffness

In the context of applications, it may be important to investigate the control problem for non-linear oscillatory systems controlled by adjustable variation, over a small interval, of some parameter characterizing the elastic restoring force, such as the coefficient of elasticity. This mode of control is extremely effective when there are pronounced oscillations, since it may lead to a rapid, e.g. exponential, change in amplitude, as in the case of parametric oscillations at resonance. Present-day technological and computational means make it possible to realize this model of control in real time [2–4].

Suppose the centre of mass of a rigid body can move along a fixed continuous curve $\mathbf{r}(q)$; the body is subject to an elastic force with adjustable coefficient $\varkappa$; the potential of the force is $W(r, \varkappa)$; we have

$$\begin{gathered} K = \tfrac{1}{2} a^2(q)\dot q^2, \ \ W = V(q,\varkappa), \ \ \varkappa = \varkappa_0(1+\varepsilon\mu), \ \ |\mu| \le \mu_0 \\ a^2(q)\ddot q + a(q)a'(q)\dot q^2 + V_q'(q,\varkappa) = 0, \ \ E = \tfrac{1}{2} a^2(q)\dot q^2 + V(q,\varkappa_0) \\ \dot q = pa^{-2}(q), \ \ \dot p = p^2 a' a^{-3}(q) - V_q'(q,\varkappa), \ \ q^*(\varkappa_0) = \arg_q V_q'(q,\varkappa_0) \\ H = \tfrac{1}{2} p^2 a^{-2}(q) + V(q,\varkappa), \ \ E = \tfrac{1}{2} p^2 a^{-2}(q) + V(q,\varkappa_0) \end{gathered} \tag{4.11}$$

Without loss of generality, we can assume that the mass m of the body and the parameters $\varkappa_0$ and μ_0 are equal to unity. The case in which $V(q, \varkappa)$ is a multiplicative function of $\varkappa$, that is, $V_q' = \varkappa U'(q)$, where U is a smooth function of q, is of special interest. Then the point of rest q^* is independent of the parameter $\varkappa_0$ and we may assume that $q^* = 0$; we shall assume that it is stable. We obtain an equation and boundary conditions for the unperturbed energy integral E (4.11) with $\varepsilon > 0$, in the first approximation with respect to ε

$$\begin{gathered} \dot E = \varepsilon F(q)\dot q\mu, \ \ F = -V_{q\varkappa}''(q,1) \ \ (F = -U_q'(q)) \\ E(0) = E^0, \ \ E(t_f) = E^f, \ \ t_f \to \min_\mu, \ \ |\mu| \le 1 \end{gathered} \tag{4.12}$$

The solution of problem (4.12) is constructed approximately, following the scheme described previously; we have the following formulae

$$\begin{gathered} \mu^* = \operatorname{sign}(\Delta E F\dot q), \ \ \dot q = \pm\sqrt{2}a^{-1}(E - V)^{1/2}, \ \ \tau_f = \tau(E^f, E^0, E^f) \\ \tau = \tau(E, E^0, E^f) = \operatorname{sign}\Delta E^0 \int_{E^0}^{E} \frac{d\chi}{\Phi(\chi)}, \ \ \Phi(E) = \frac{1}{T}\int_0^T |F\dot q|\, dt = \frac{4E}{T(E)} \end{gathered} \tag{4.13}$$

By the third relation in (4.13), the quantity τ_f will be finite when $E^{0,f} = 0$ if the period satisfies an estimate $T(E) \sim E^\gamma$, $\gamma > 0$, that is, the oscillation frequency increases without limit as $E \to 0$. Let $V = \varkappa|q|^\rho$; then, by (4.4), we obtain

$$T(E) = \theta(\varkappa,\rho)E^{1/\rho-1/2}, \ \ \Phi(E) = (4/\theta)E^{-1/\rho+3/2} \tag{4.14}$$

Formulae (4.13) and (4.14) imply the controllability condition $\rho < 2$; this means that, in the case of linear elasticity (and the potential is a power function of the displacement, to a higher order), the required time τ_f is not bounded as $E^{0,f} \to 0$. The required functions $E(\tau)$ and τ_f are explicitly defined (see (4.4)). Note that the optimal control μ^* changes sign four times in a time interval t equal to the period, since sign $(F(q)q) = 1$. If the behaviour of the potential $V(q, 1)$ is more complicated, system (4.11) may have several equilibria. This situation requires further study of the control problem, because of the presence of saddle points and separatrix-crossing by the phase trajectory.

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